Probability distribution of the interface width in surface roughening: analogy with a Levy flight

1991 J. Phys. A: Math. Gen. 24 L925

(http://iopscience.iop.org/0305-4470/24/16/008)

J. Phys. A: Math. Gen. 24 (1991) L925–L931. Printed in the UK

LETTER TO THE EDITOR

Probability distribution of the interface width in surface roughening: analogy with a Lévy flight

Shlomo Havlin†‡, Sergey V Buldyrev†, H Eugene Stanley† and George H Weiss‡

† Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215, USA
‡ Division of Computer Research and Technology, National Institutes of Health, Bethesda, MD 20205, USA

Received 11 June 1991

Abstract. We present a closed-form expression for the *probability distribution of the height fluctuations* in the Zhang model of anomalous surface roughening. The result—which includes both the steady state behaviour and the time evolution to the steady state—is based on an analogy between the ($d = 1 + 1$)-dimensional 'surface' problem and a $d = 1$ Lévy flight. In the limit case of conventional ballistic deposition we obtain a Gaussian distribution for the height fluctuations. Our results are corroborated by detailed numerical simulations.

Surface roughening and Lévy flights are two subjects of intensive current experimental and theoretical interest (for reviews see [1–3].) These two topics in statistical physics are generally treated as separate problems. In this letter we present a theory for surface roughening that connects this phenomenon in $1 + 1$ dimensions to the properties of a simple one-dimensional Lévy flight.

Various surface growth models such as the Kardar–Parisi–Zhang (KPZ) equation and ballistic deposition are consistent with a scaling form for the RMS surface width [4,5],

$$w(L,t) \sim L^{\alpha} f(t/L^{\alpha/\beta}) \tag{1}$$

with the property that for $1 \ll t \ll t_{\times}$, $w \sim t^{\beta}$ and for $t \gg t_{\times}$, $w \sim L^{\alpha}$. Here $t_{\times} \sim L^{z}$ is the crossover time to a 'steady state' in which the width is time-independent, L is the linear size of the system, and $z \equiv \alpha/\beta = 2 - \alpha$ is the dynamic exponent. An analysis of the KPZ equation and numerical results on ballistic deposition yield $\alpha = 1/2$ and $\beta = 1/3$ for $d = 1$. However, recent experiments on surface roughening suggest anomalously large values for both α and β: $\alpha \cong 0.75$ and $\beta \cong 0.6$ [6–8].

Recently Zhang [9] suggested that the anomalous roughening found in experiments can be explained by an uncorrelated 'noise' $\eta(r,t)$ obeying a power-law distribution, $p(\eta) \sim \eta^{-\mu-1}$ where $\eta \geqslant 1$. This anomalous noise can be simulated in a deposition context by depositing rods of size ℓ sampled from a power-law probability distribution,

$$p(\ell) \sim \ell^{-\mu-1}. \tag{2}$$

The growth rule is similar to the conventional ballistic deposition rule, i.e., a deposited rod is attached to the highest nearest-neighbour surface site. The site at which deposition next occurs can be chosen either *deterministically* as in [9, 10] or randomly as in [11]. Here we use the same growth rule as in [11], and a 48-bit random number generator which was tested by verifying that it explicitly reproduced the probability distribution (2). The Zhang idea [9] has received recent support from both theory [12, 13] and numerical simulation [9–11], which suggest that both exponents α and β are anomalously large, and depend continuously on the parameter μ (at least for $\mu < \mu_c \approx 5$).

Here we derive a closed-form expression for the probability distribution for the fluctuations $\delta h(r,t) \equiv h(r,t) - \langle h(t) \rangle$ in surface height $h(r,t)$ at point r. Very recently, this distribution and its spatial scaling properties were studied numerically [10]; a power-law tail was found, but no explanation of its origin was given. To this end, we develop a formal analogy between anomalous roughening and the statistics of a Lévy walk [2]; at each unit of time a random walker steps a unit length, moving ℓ successive steps in the same direction before randomly changing direction, with ℓ taken from a Lévy distribution $p(\ell) \sim \ell^{-\mu-1}$. Since for $\mu > 1$, $\langle \ell \rangle$ is finite, the Lévy walk model will have the same distribution as the Lévy flight [2] where the walker makes ℓ steps in one unit of time. The probability density $P(r,n)$ that the walker is at position r after n steps has a tail distribution of the form [14, 15]

$$P(r,n) \sim n/r^{\mu+1} = \frac{1}{n^{1/\mu}} \left(\frac{r}{n^{1/\mu}} \right)^{-\mu-1} \qquad [r \gg r_\times] \tag{3a}$$

where $r_\times \sim n^{1/2}$ for $\mu > 2$ and $r_\times \sim n^{1/\mu}$ for $\mu \leqslant 2$.

Consider first $t = 1$, which corresponds by definition to the deposition of L rods—each of length ℓ_i where ℓ_i is a random variable chosen from (2). Since the lengths of the rods obey a Lévy distribution, so also the set of differences in the lengths of the rods (which is the set of numbers $\delta h(r, t = 1)$) must obey a Lévy distribution. Hence the probability density $P(\delta h, L, t = 1)$ for δh is exactly the distribution of a Lévy walk, and from (3a)

$$P(\delta h, L, t = 1) \sim \frac{1}{L^{1/\mu}} \left(\frac{\delta h}{L^{1/\mu}} \right)^{-\mu-1} \qquad [\delta h \gg L^{1/\mu}]. \tag{3b}$$

Now consider time $t = 2$, corresponding to two layers of rods. The distribution of the sum of two rods is also a Lévy distribution with L scaled by a factor of two [2]. Hence equation (3b) holds provided L is replaced by $2L$.

For the simplest 'random deposition' model, equation (3b) still holds for any t, with L replaced by N, where $N \approx tL$ is the total number of deposited rods. For the Zhang variant of ballistic deposition, we assume that the distribution does not change—only that time and space are now correlated due to the growth rule or the KPZ equation. Therefore a surface fluctuation can grow laterally only until a steady state is achieved, at time $t_\times \sim L^z$. For $t \gg t_\times$, the behaviour is quite different: equation (3*b*) will hold but with L replaced not by tL but rather by $t_\times L \equiv L^{z+1} \equiv L^{3-\alpha}$. The probability distribution becomes independent of time, assuming the asymptotic form

$$P(\delta h, L, t \gg t_\times) \sim \frac{1}{L^{(3-\alpha)/\mu}} \left(\frac{\delta h}{L^{(3-\alpha)/\mu}}\right)^{-\mu-1} \qquad [\delta h \gg L^{(3-\alpha)/\mu}]. \tag{4}$$

Since for $t \gg t_\times$, $\delta h \sim w \sim L^\alpha$, we find from the scaling form of (4) a self-consistent equation for α, $\alpha = (3-\alpha)/\mu$ or

$$\alpha = \frac{3}{\mu+1} \qquad [\mu > 2]. \tag{5}$$

Equation (5) was derived as a lower bound by Zhang and Krug [12, 13]. Note that α assumes its classical value $\alpha = 1/2$ at $\mu = \mu_c = 5$. Substituting (5) into (4), we obtain the scaling form

$$P(\delta h, L, t \gg t_\times) \sim \frac{1}{L^{3/(\mu+1)}} \left(\frac{\delta h}{L^{3/(\mu+1)}}\right)^{-\mu-1}. \tag{6}$$

To obtain the probability at early time $1 \ll t \ll t_\times$, we use again the time-space relation $t_\times \sim L^z$ of (1). Thus L in (3*b*) should be replaced by $N = tL = tt^{1/z}$ yielding

$$P(\delta h, t) \sim \frac{1}{t^{(z+1)/\mu z}} \left(\frac{\delta h}{t^{(z+1)/\mu z}}\right)^{-\mu-1}. \tag{7}$$

Since $\delta h \sim t^\beta$ we obtain,

$$\beta = \frac{z+1}{\mu z} = \frac{3}{2\mu - 1} \qquad [\mu > 2]. \tag{8}$$

The above considerations, equations (4)–(8), are valid for $\mu > 2$. For $\mu \leqslant 2$, the relation $t_\times L \sim L^{z+1}$ fails since $t_\times$ is bounded from below by L. A 'rare' fluctuation produced by a very long rod cannot disappear faster than the time it takes for the 'tree growth' appearing on the top of that rod to overlap the system. The horizontal width of such a 'tree' typically grows by one site in each time step, so $t_\times \geqslant L$. Thus one must repeat the arguments leading to equations (4)–(8) using $t_\times L = L^2$ from which follows

$$\alpha = \beta = \frac{2}{\mu} \qquad [\mu < 2]. \tag{9}$$

Note that (9) complements the Zhang–Krug prediction for values of μ below 2. From (1) we calculated α and β; the results, plotted in figure 1, are compared with the prediction given by equations (5), (8) and (9).

The analogy to Lévy walks predicts not only the tails of the probability densities but also their behaviour in the range of small fluctuations. In this range, the distribution of Lévy walks $P(r, n)$ is known to be Gaussian [14, 15], predicting that $P(\delta h, L, t)$

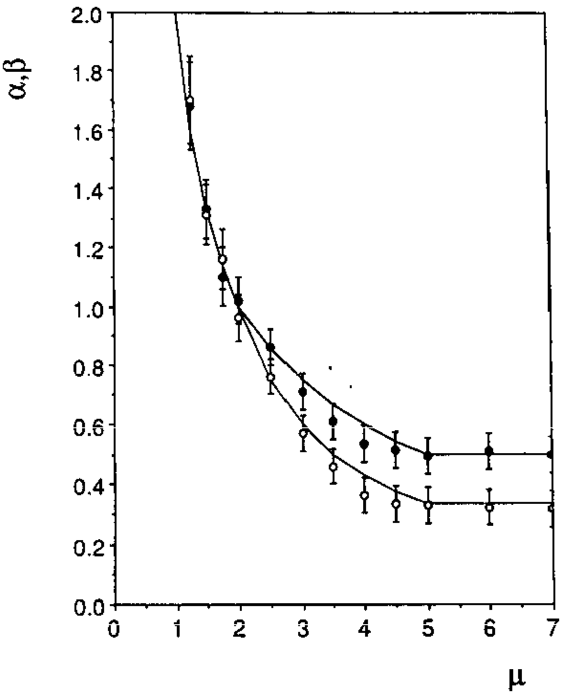


Figure 1. Comparison of numerical results for exponents α (●) and β (○) with theoretical predictions given for $\mu \geqslant 2$ by equations (5) (solid line) and (8) (dotted line), and for $\mu \leqslant 2$ by (9) (solid line). The results for $\mu \geqslant 2$ are taken from Buldyrev *et al* [11].

is also Gaussian. Thus equations (6) and (7)—as well as the crossover from Gaussian to power law—can be combined to a single scaling relation

$$P(\delta h, L, t) \sim \frac{1}{w} F\left(\frac{\delta h}{w}\right). \tag{10a}$$

Here $w = w(L,t)$, is the *first* moment of $P(\delta h, L, t)$; our numerical results for w support (1). The scaling function in (10a) obeys

$$F(y) \sim \begin{cases} \exp(-ay^2) & [y < y_\times] \\ y^{-\mu-1} & [y > y_\times]. \end{cases} \tag{10b}$$

The crossover from Gaussian to a power law occurs at a value $y_\times$ of $y \equiv \delta h/w$ which increases as μ increases as expected from the analysis in [14, 15]. We use in our simulations the first moment of $P(\delta h, L, t)$ because it converges in the whole range of studied values of μ while the second moment diverges for $\mu \leqslant 2$. For $\mu > 2$ both moments have the same behaviour.

The data collapse shown in figure 2 supports (10). Moreover, plotting the data in figure 3(a) as $\log\log[P(\delta h, L)/P(0, L)]$ versus $\log \delta h$ for several values of L shows a clear range of slope 2, supporting a Gaussian form for small δh. Some deviations from scaling form given by (10) may be due to finite-size corrections to scaling or to multifractality at small distances found by Barabási *et al* [16].

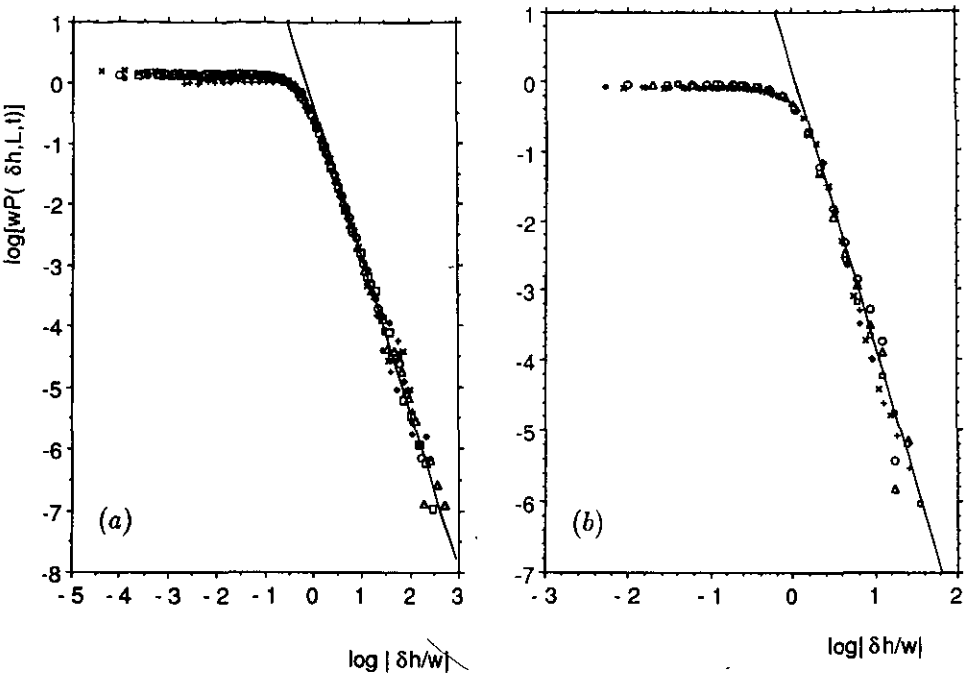


Figure 2. Log–log scaling plot of $wP(\delta h, L, t)$ against $|\delta h|/w$. Here w is the *first* moment of $P(\delta h, L, t)$ for (a) $\mu = 1.5$ and (b) $\mu = 3$. Symbols in (a) are: + ($L = 1028, t = 64$); ◇ ($L = 2048, t = 256$); × ($L = 2048, t = 1024$); △($L = 256, t > 4096$);□ ($L = 512, t > 4096$); ○ ($L = 1024, t > 4096$). Symbols in (b) are: □ ($t = 32, L = 2048$); △ ($t = 128, L = 2048$); ○ ($t = 512, L = 2048$); + ($t > 16384, L = 512$); × ($t > 16384, L = 1024$); ◇ ($t > 32768, L = 2048$). The straight lines have slopes of (a) 2.5 and (b) 4, as predicted by (6).

It can be seen from figure 3(b) that as μ increases, the Gaussian region of the probability distribution also increases. Thus we expect that in the case of conventional ballistic deposition ($\mu = \infty$) the distribution will be Gaussian:

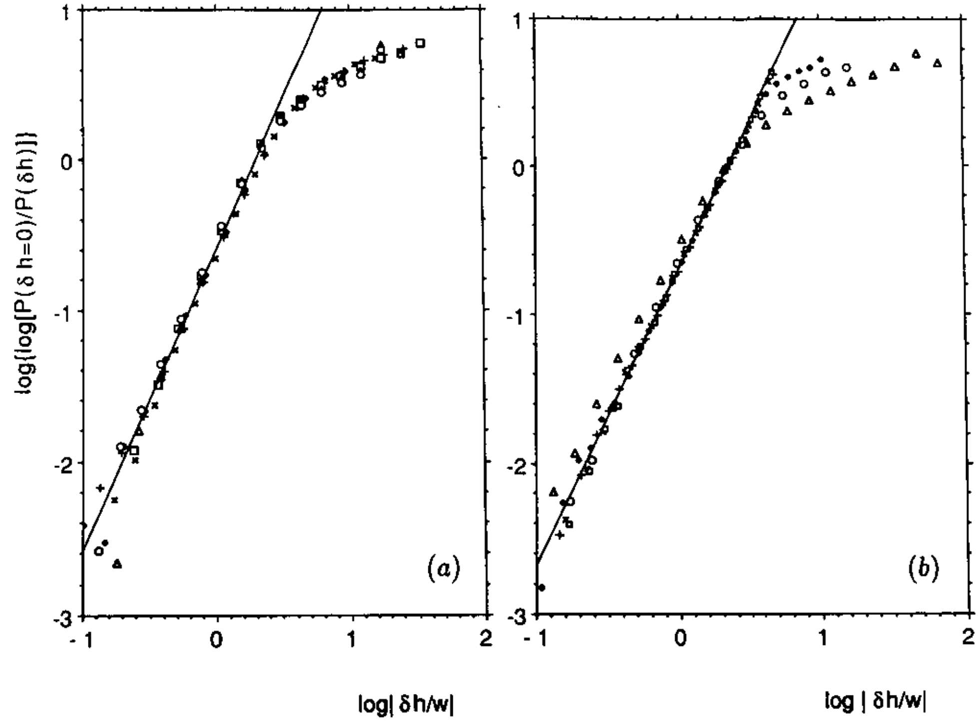
$$P(\delta h, t) \sim \frac{1}{t^{1/3}} \exp[-a(\delta h)^2/t^{2/3}] \qquad [t \ll t_\times] \tag{11a}$$

and

$$P(\delta h, L) \sim \frac{1}{L^{1/2}} \exp[-a(\delta h)^2/L] \qquad [t \gg t_\times \sim L^{1/z}]. \tag{11b}$$

In figure 3(c) we present numerical data supporting (11).

In summary, we present an analogy between the random surface problem in $1+1$ dimensions and the simple $d = 1$ Lévy walk. The analogy predicts an *analytic form* for the probability density for the height fluctuations in anomalous surface roughening. The probability density crosses over from a Gaussian for small δh to an asymptotic *power-law* Lévy distribution for large δh. For the conventional ballistic deposition model, this analogy is with a conventional random walk with unit step lengths—and the probability distribution is Gaussian. Our approach suggests that for $\mu \leqslant 2$ a new singular class of surfaces does exist. We cannot rule out the possibility that the



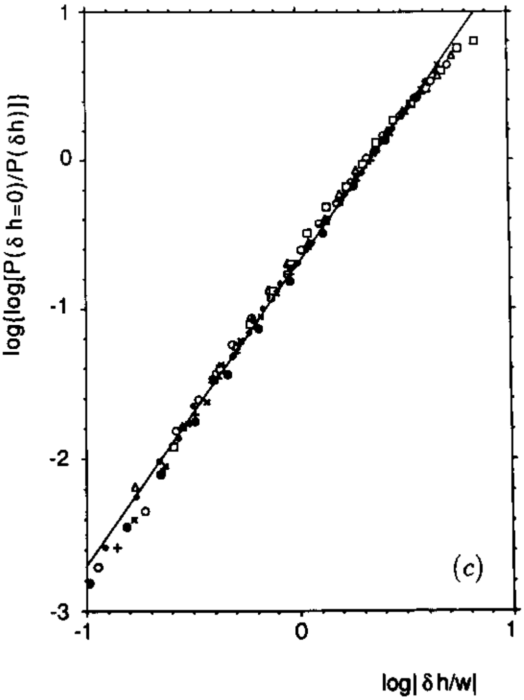


Figure 3. (a) Log–log scaling plots of $\log[P(\delta h = 0, L, t)/P(\delta h, L, t)]$ versus $|\delta h|/w$ for $\mu = 3$ for different system sizes L and times t. The straight line has the slope 2, as should be for Gaussian distribution. The symbols are the same as those used in figure 2(b). (b) Log–log plot of $\log[P(\delta h = 0, L, t)/P(\delta h, L, t)]$ as a function of $|\delta h|/w$ for different values of μ (the data for each value of μ obtained for $L = 1024$ and large t, greater than $t_\times$): □ ($\mu = \infty$); × ($\mu = 6$); + ($\mu = 5$); ◊ ($\mu = 4$); ○ ($\mu = 3$); △ ($\mu = 2$). The crossover value of $\delta h/w$ at which the behaviour changes from Gaussian (straight line with slope 2) to power law increases gradually with the value of μ. (c) Log–log plot of $\log[P(\delta h = 0, L, t)/P(\delta h, L, t)]$ as a function of $|\delta h|/w$ for $\mu = \infty$, which is the case of *conventional* ballistic deposition. Here w scales according to (1) with $\alpha = 1/2$, $\beta = 1/3$. The Gaussian behaviour is found in the entire range of δh: □ ($t = 64, L = 4096$); △ ($t = 256, L = 4096$); ○ ($t = 1024, L = 4096$); + ($L = 512, t > 16384$); × ($L = 1024, t > 16384$); ◊ ($L = 2048, t > 32768$); ● ($L = 4096, t > 32768$).

same scaling form of distribution (10) is also valid for the Zhang model in higher dimensions [17], although the surface itself can no longer be considered as a record of a linear walk or a flight.

We wish to thank NSF, ONR, and BSF for support, and J Kertész and T Vicsek for valuable discussions.

References

[1] Meakin P 1987 *CRC Critical Rev. Solid State Mater. Sci.* **13** 143
[2] Shlesinger M F 1988 *Ann. Rev. Phys. Chem.* **39** 269
[3] Ott A, Bouchaud J P, Langevin D and Urbach W 1990 *Phys. Rev. Lett.* **65** 2201
[4] Kardar M, Parisi G and Zhang Y-C 1986 *Phys. Rev. Lett.* **56** 889
[5] Family F and Vicsek T 1985 *J. Phys. A: Math. Gen.* **18** L75
[6] Rubio M A, Edwards C A, Dougherty A and Gollub J P 1989 *Phys. Rev. Lett.* **63** 1685
[7] Horváth V K, Family F and Vicsek T 1991 *J. Phys. A: Math. Gen.* **24** L25
[8] Vicsek T, Cserzö M and Horváth V K 1990 *Physica* **167A** 315
[9] Zhang Y-C 1990 *J. Physique* **51** 2113
[10] Amar J G and Family F 1991 *J. Phys. A: Math. Gen.* **24** L79
[11] Buldyrev S V, Havlin S, Kertész J, Stanley H E and Vicsek T 1991 *Phys. Rev.* A **43** 7113
[12] Zhang Y-C 1990 *Physica* **170** 1
[13] Krug J 1991 *J. Physique* I **1** 9
[14] Zumofen G, Klafter J and Blumen A 1990 *Chem. Phys.* **146** 433
[15] Araujo M, Havlin S, Weiss G H and Stanley H E 1991 *Phys. Rev.* A **43** 5207
[16] Barabási A-L, Bourbonnais R, Jensen M, Kertész J, Vicsek T and Zhang Y-C, unpublished
[17] Bourbonnais R, Kertész J and Wolf D E 1991 *J. Physique* II **1** 493